PG No. 10 S_4 $\bar{4}$ [tetragonal] (polar, internal polar dipole)

* Harmonics for rank 0

$\vec{\mathbb{Q}}_0^{(1,1)}[q](A)$

** symmetry

$$1$$

** expression

$$\frac{\sqrt{3}Q_x x}{3} + \frac{\sqrt{3}Q_y y}{3} + \frac{\sqrt{3}Q_z z}{3}$$

* Harmonics for rank 1

$\vec{\mathbb{Q}}_1^{(1,-1)}[q](B)$

** symmetry

$$z$$

** expression

$$Q_z$$

$\vec{\mathbb{Q}}_1^{(1,1)}[q](B)$

** symmetry

$$z$$

** expression

$$\frac{3\sqrt{10}Q_x xz}{10} + \frac{3\sqrt{10}Q_y yz}{10} - \frac{\sqrt{10}Q_z\left(x^2+y^2-2z^2\right)}{10}$$

$\vec{\mathbb{Q}}_{1,1}^{(1,-1)}[q](E)$, $\vec{\mathbb{Q}}_{1,2}^{(1,-1)}[q](E)$

** symmetry

$$x$$

$$y$$

** expression

$$Q_x$$

$$Q_y$$

$\vec{\mathbb{Q}}_{1,1}^{(1,1)}[q](E)$, $\vec{\mathbb{Q}}_{1,2}^{(1,1)}[q](E)$

** symmetry

$$x$$

$$y$$

** expression

$$\frac{\sqrt{10}Q_x\left(2x^2-y^2-z^2\right)}{10} + \frac{3\sqrt{10}Q_y xy}{10} + \frac{3\sqrt{10}Q_z xz}{10}$$

$$\frac{3\sqrt{10}Q_x xy}{10} - \frac{\sqrt{10}Q_y\left(x^2-2y^2+z^2\right)}{10} + \frac{3\sqrt{10}Q_z yz}{10}$$

* Harmonics for rank 2

$\vec{\mathbb{Q}}_2^{(1,-1)}[q](A)$

** symmetry

$$-\frac{x^2}{2} - \frac{y^2}{2} + z^2$$

** expression

$$-\frac{\sqrt{6}Q_x x}{6} - \frac{\sqrt{6}Q_y y}{6} + \frac{\sqrt{6}Q_z z}{3}$$

$\vec{\mathbb{Q}}_2^{(1,1)}[q](A)$

** symmetry

$$-\frac{x^2}{2} - \frac{y^2}{2} + z^2$$

** expression

$$-\frac{\sqrt{21}Q_x x\left(x^2+y^2-4z^2\right)}{14}-\frac{\sqrt{21}Q_y y\left(x^2+y^2-4z^2\right)}{14}-\frac{\sqrt{21}Q_z z\left(3x^2+3y^2-2z^2\right)}{14}$$

$\vec{\mathbb{Q}}_2^{(1,-1)}[q](B,1)$
** symmetry

$$\frac{\sqrt{3}\left(x-y\right)\left(x+y\right)}{2}$$

** expression

$$\frac{\sqrt{2}Q_x x}{2}-\frac{\sqrt{2}Q_y y}{2}$$

$\vec{\mathbb{Q}}_2^{(1,-1)}[q](B,2)$
** symmetry

$$\sqrt{3}xy$$

** expression

$$\frac{\sqrt{2}Q_x y}{2}+\frac{\sqrt{2}Q_y x}{2}$$

$\vec{\mathbb{Q}}_2^{(1,1)}[q](B,1)$
** symmetry

$$\frac{\sqrt{3}\left(x-y\right)\left(x+y\right)}{2}$$

** expression

$$\frac{\sqrt{7}Q_x x\left(3x^2-7y^2-2z^2\right)}{14}+\frac{\sqrt{7}Q_y y\left(7x^2-3y^2+2z^2\right)}{14}+\frac{5\sqrt{7}Q_z z\left(x-y\right)\left(x+y\right)}{14}$$

$\vec{\mathbb{Q}}_2^{(1,1)}[q](B,2)$
** symmetry

$$\sqrt{3}xy$$

** expression

$$\frac{\sqrt{7}Q_x y\left(4x^2-y^2-z^2\right)}{7}-\frac{\sqrt{7}Q_y x\left(x^2-4y^2+z^2\right)}{7}+\frac{5\sqrt{7}Q_z xyz}{7}$$

$\vec{\mathbb{Q}}_{2,1}^{(1,-1)}[q](E)$, $\vec{\mathbb{Q}}_{2,2}^{(1,-1)}[q](E)$
** symmetry

$$\sqrt{3}yz$$

$$\sqrt{3}xz$$

** expression

$$\frac{\sqrt{2}Q_y z}{2}+\frac{\sqrt{2}Q_z y}{2}$$

$$\frac{\sqrt{2}Q_x z}{2}+\frac{\sqrt{2}Q_z x}{2}$$

$\vec{\mathbb{Q}}_{2,1}^{(1,1)}[q](E)$, $\vec{\mathbb{Q}}_{2,2}^{(1,1)}[q](E)$
** symmetry

$$\sqrt{3}yz$$

$$\sqrt{3}xz$$

** expression

$$\frac{5\sqrt{7}Q_x xyz}{7}-\frac{\sqrt{7}Q_y z\left(x^2-4y^2+z^2\right)}{7}-\frac{\sqrt{7}Q_z y\left(x^2+y^2-4z^2\right)}{7}$$

$$\frac{\sqrt{7}Q_x z\left(4x^2-y^2-z^2\right)}{7}+\frac{5\sqrt{7}Q_y xyz}{7}-\frac{\sqrt{7}Q_z x\left(x^2+y^2-4z^2\right)}{7}$$

* Harmonics for rank 3

$\vec{\mathbb{Q}}_3^{(1,-1)}[q](A,1)$
** symmetry

$$\sqrt{15}xyz$$

** expression

$$Q_x yz + Q_y xz + Q_z xy$$

$\vec{\mathbb{Q}}_3^{(1,-1)}[q](A,2)$
** symmetry

$$\frac{\sqrt{15}z\left(x-y\right)\left(x+y\right)}{2}$$

** expression

$$Q_x xz - Q_y yz + \frac{Q_z\left(x-y\right)\left(x+y\right)}{2}$$

$\vec{\mathbb{Q}}_3^{(1,1)}[q](A,1)$
** symmetry

$$\sqrt{15}xyz$$

** expression

$$\frac{\sqrt{15}Q_x yz\left(6x^2-y^2-z^2\right)}{6} - \frac{\sqrt{15}Q_y xz\left(x^2-6y^2+z^2\right)}{6} - \frac{\sqrt{15}Q_z xy\left(x^2+y^2-6z^2\right)}{6}$$

$\vec{\mathbb{Q}}_3^{(1,1)}[q](A,2)$
** symmetry

$$\frac{\sqrt{15}z\left(x-y\right)\left(x+y\right)}{2}$$

** expression

$$\frac{\sqrt{15}Q_x xz\left(5x^2-9y^2-2z^2\right)}{12} + \frac{\sqrt{15}Q_y yz\left(9x^2-5y^2+2z^2\right)}{12} - \frac{\sqrt{15}Q_z\left(x-y\right)\left(x+y\right)\left(x^2+y^2-6z^2\right)}{12}$$

$\vec{\mathbb{Q}}_3^{(1,-1)}[q](B)$
** symmetry

$$-\frac{z\left(3x^2+3y^2-2z^2\right)}{2}$$

** expression

$$-\frac{\sqrt{15}Q_x xz}{5} - \frac{\sqrt{15}Q_y yz}{5} - \frac{\sqrt{15}Q_z\left(x^2+y^2-2z^2\right)}{10}$$

$\vec{\mathbb{Q}}_3^{(1,1)}[q](B)$
** symmetry

$$-\frac{z\left(3x^2+3y^2-2z^2\right)}{2}$$

** expression

$$-\frac{5Q_x xz\left(3x^2+3y^2-4z^2\right)}{12} - \frac{5Q_y yz\left(3x^2+3y^2-4z^2\right)}{12} + \frac{Q_z\left(3x^4+6x^2y^2-24x^2z^2+3y^4-24y^2z^2+8z^4\right)}{12}$$

$\vec{\mathbb{Q}}_{3,1}^{(1,-1)}[q](E,1)$, $\vec{\mathbb{Q}}_{3,2}^{(1,-1)}[q](E,1)$
** symmetry

$$\frac{x\left(2x^2-3y^2-3z^2\right)}{2}$$

$$-\frac{y\left(3x^2-2y^2+3z^2\right)}{2}$$

** expression

$$\frac{\sqrt{15}Q_x\left(2x^2-y^2-z^2\right)}{10} - \frac{\sqrt{15}Q_y xy}{5} - \frac{\sqrt{15}Q_z xz}{5}$$

$$-\frac{\sqrt{15}Q_x xy}{5} - \frac{\sqrt{15}Q_y\left(x^2-2y^2+z^2\right)}{10} - \frac{\sqrt{15}Q_z yz}{5}$$

$\vec{\mathbb{Q}}_{3,1}^{(1,-1)}[q](E,2)$, $\vec{\mathbb{Q}}_{3,2}^{(1,-1)}[q](E,2)$
** symmetry

$$-\frac{\sqrt{15}y\left(x-z\right)\left(x+z\right)}{2}$$

$$\frac{\sqrt{15}x\left(y-z\right)\left(y+z\right)}{2}$$

** expression

$$-Q_{x}xy-\frac{Q_{y}\left(x-z\right)\left(x+z\right)}{2}+Q_{z}yz$$

$$\frac{Q_{x}\left(y-z\right)\left(y+z\right)}{2}+Q_{y}xy-Q_{z}xz$$

$\vec{\mathbb{Q}}_{3,1}^{(1,1)}[q](E,1)$, $\vec{\mathbb{Q}}_{3,2}^{(1,1)}[q](E,1)$
** symmetry

$$\frac{x\left(2x^{2}-3y^{2}-3z^{2}\right)}{2}$$

$$-\frac{y\left(3x^{2}-2y^{2}+3z^{2}\right)}{2}$$

** expression

$$\frac{Q_{x}\left(8x^{4}-24x^{2}y^{2}-24x^{2}z^{2}+3y^{4}+6y^{2}z^{2}+3z^{4}\right)}{12}+\frac{5Q_{y}xy\left(4x^{2}-3y^{2}-3z^{2}\right)}{12}+\frac{5Q_{z}xz\left(4x^{2}-3y^{2}-3z^{2}\right)}{12}$$

$$-\frac{5Q_{x}xy\left(3x^{2}-4y^{2}+3z^{2}\right)}{12}+\frac{Q_{y}\left(3x^{4}-24x^{2}y^{2}+6x^{2}z^{2}+8y^{4}-24y^{2}z^{2}+3z^{4}\right)}{12}-\frac{5Q_{z}yz\left(3x^{2}-4y^{2}+3z^{2}\right)}{12}$$

$\vec{\mathbb{Q}}_{3,1}^{(1,1)}[q](E,2)$, $\vec{\mathbb{Q}}_{3,2}^{(1,1)}[q](E,2)$
** symmetry

$$-\frac{\sqrt{15}y\left(x-z\right)\left(x+z\right)}{2}$$

$$\frac{\sqrt{15}x\left(y-z\right)\left(y+z\right)}{2}$$

** expression

$$-\frac{\sqrt{15}Q_{x}xy\left(5x^{2}-2y^{2}-9z^{2}\right)}{12}+\frac{\sqrt{15}Q_{y}\left(x-z\right)\left(x+z\right)\left(x^{2}-6y^{2}+z^{2}\right)}{12}-\frac{\sqrt{15}Q_{z}yz\left(9x^{2}+2y^{2}-5z^{2}\right)}{12}$$

$$\frac{\sqrt{15}Q_{x}\left(y-z\right)\left(y+z\right)\left(6x^{2}-y^{2}-z^{2}\right)}{12}-\frac{\sqrt{15}Q_{y}xy\left(2x^{2}-5y^{2}+9z^{2}\right)}{12}+\frac{\sqrt{15}Q_{z}xz\left(2x^{2}+9y^{2}-5z^{2}\right)}{12}$$

* Harmonics for rank 4
$\vec{\mathbb{Q}}_{4}^{(1,-1)}[q](A,1)$
** symmetry

$$\frac{\sqrt{21}\left(x^{4}-3x^{2}y^{2}-3x^{2}z^{2}+y^{4}-3y^{2}z^{2}+z^{4}\right)}{6}$$

** expression

$$\frac{\sqrt{3}Q_{x}x\left(2x^{2}-3y^{2}-3z^{2}\right)}{6}-\frac{\sqrt{3}Q_{y}y\left(3x^{2}-2y^{2}+3z^{2}\right)}{6}-\frac{\sqrt{3}Q_{z}z\left(3x^{2}+3y^{2}-2z^{2}\right)}{6}$$

$\vec{\mathbb{Q}}_{4}^{(1,-1)}[q](A,2)$
** symmetry

$$-\frac{\sqrt{15}\left(x^{4}-12x^{2}y^{2}+6x^{2}z^{2}+y^{4}+6y^{2}z^{2}-2z^{4}\right)}{12}$$

** expression

$$-\frac{\sqrt{105}Q_{x}x\left(x^{2}-6y^{2}+3z^{2}\right)}{42}+\frac{\sqrt{105}Q_{y}y\left(6x^{2}-y^{2}-3z^{2}\right)}{42}-\frac{\sqrt{105}Q_{z}z\left(3x^{2}+3y^{2}-2z^{2}\right)}{42}$$

$\vec{\mathbb{Q}}_{4}^{(1,-1)}[q](A,3)$

** symmetry

$$\frac{\sqrt{35}xy\left(x-y\right)\left(x+y\right)}{2}$$

** expression

$$\frac{\sqrt{5}Q_{x}y\left(3x^{2}-y^{2}\right)}{4}+\frac{\sqrt{5}Q_{y}x\left(x^{2}-3y^{2}\right)}{4}$$

$\vec{\mathbb{Q}}_{4}^{(1,1)}[q](A,1)$
** symmetry

$$\frac{\sqrt{21}\left(x^{4}-3x^{2}y^{2}-3x^{2}z^{2}+y^{4}-3y^{2}z^{2}+z^{4}\right)}{6}$$

** expression

$$\frac{\sqrt{1155}Q_{x}x\left(x^{4}-5x^{2}y^{2}-5x^{2}z^{2}+3y^{4}-3y^{2}z^{2}+3z^{4}\right)}{66}+\frac{\sqrt{1155}Q_{y}y\left(3x^{4}-5x^{2}y^{2}-3x^{2}z^{2}+y^{4}-5y^{2}z^{2}+3z^{4}\right)}{66}+\frac{\sqrt{1155}Q_{z}z\left(3x^{4}-3x^{2}y^{2}-5x^{2}z^{2}+3y^{4}-5y^{2}z^{2}+z^{4}\right)}{66}$$

$\vec{\mathbb{Q}}_{4}^{(1,1)}[q](A,2)$
** symmetry

$$-\frac{\sqrt{15}\left(x^{4}-12x^{2}y^{2}+6x^{2}z^{2}+y^{4}+6y^{2}z^{2}-2z^{4}\right)}{12}$$

** expression

$$-\frac{\sqrt{33}Q_{x}x\left(5x^{4}-88x^{2}y^{2}+38x^{2}z^{2}+33y^{4}+66y^{2}z^{2}-30z^{4}\right)}{132}-\frac{\sqrt{33}Q_{y}y\left(33x^{4}-88x^{2}y^{2}+66x^{2}z^{2}+5y^{4}+38y^{2}z^{2}-30z^{4}\right)}{132}+\frac{\sqrt{33}Q_{z}z\left(3x^{4}+132x^{2}y^{2}-50x^{2}z^{2}+3y^{4}-50y^{2}z^{2}+10z^{4}\right)}{132}$$

$\vec{\mathbb{Q}}_{4}^{(1,1)}[q](A,3)$
** symmetry

$$\frac{\sqrt{35}xy\left(x-y\right)\left(x+y\right)}{2}$$

** expression

$$\frac{\sqrt{77}Q_{x}y\left(6x^{4}-11x^{2}y^{2}-3x^{2}z^{2}+y^{4}+y^{2}z^{2}\right)}{22}-\frac{\sqrt{77}Q_{y}x\left(x^{4}-11x^{2}y^{2}+x^{2}z^{2}+6y^{4}-3y^{2}z^{2}\right)}{22}+\frac{9\sqrt{77}Q_{z}xyz\left(x-y\right)\left(x+y\right)}{22}$$

$\vec{\mathbb{Q}}_{4}^{(1,-1)}[q](B,1)$
** symmetry

$$\frac{\sqrt{5}\left(x-y\right)\left(x+y\right)\left(x^{2}+y^{2}-6z^{2}\right)}{4}$$

** expression

$$\frac{\sqrt{35}Q_{x}x\left(x^{2}-3z^{2}\right)}{14}-\frac{\sqrt{35}Q_{y}y\left(y^{2}-3z^{2}\right)}{14}-\frac{3\sqrt{35}Q_{z}z\left(x-y\right)\left(x+y\right)}{14}$$

$\vec{\mathbb{Q}}_{4}^{(1,-1)}[q](B,2)$
** symmetry

$$-\frac{\sqrt{5}xy\left(x^{2}+y^{2}-6z^{2}\right)}{2}$$

** expression

$$-\frac{\sqrt{35}Q_{x}y\left(3x^{2}+y^{2}-6z^{2}\right)}{28}-\frac{\sqrt{35}Q_{y}x\left(x^{2}+3y^{2}-6z^{2}\right)}{28}+\frac{3\sqrt{35}Q_{z}xyz}{7}$$

$\vec{\mathbb{Q}}_{4}^{(1,1)}[q](B,1)$
** symmetry

$$\frac{\sqrt{5}\left(x-y\right)\left(x+y\right)\left(x^{2}+y^{2}-6z^{2}\right)}{4}$$

** expression

$$\frac{\sqrt{11}Q_{x}x\left(5x^{4}-4x^{2}y^{2}-46x^{2}z^{2}-9y^{4}+66y^{2}z^{2}+12z^{4}\right)}{44}+\frac{\sqrt{11}Q_{y}y\left(9x^{4}+4x^{2}y^{2}-66x^{2}z^{2}-5y^{4}+46y^{2}z^{2}-12z^{4}\right)}{44}+\frac{21\sqrt{11}Q_{z}z\left(x-y\right)\left(x+y\right)\left(x^{2}+y^{2}-2z^{2}\right)}{44}$$

$\vec{\mathbb{Q}}_4^{(1,1)}[q](B,2)$
** symmetry

$$-\frac{\sqrt{5}xy\left(x^2+y^2-6z^2\right)}{2}$$

** expression

$$-\frac{\sqrt{11}Q_xy\left(6x^4+5x^2y^2-51x^2z^2-y^4+5y^2z^2+6z^4\right)}{22}+\frac{\sqrt{11}Q_yx\left(x^4-5x^2y^2-5x^2z^2-6y^4+51y^2z^2-6z^4\right)}{22}-\frac{21\sqrt{11}Q_zxyz\left(x^2+y^2-2z^2\right)}{22}$$

$\vec{\mathbb{Q}}_{4,1}^{(1,-1)}[q](E,1)$, $\vec{\mathbb{Q}}_{4,2}^{(1,-1)}[q](E,1)$
** symmetry

$$-\frac{\sqrt{35}xz\left(x-z\right)\left(x+z\right)}{2}$$

$$\frac{\sqrt{35}yz\left(y-z\right)\left(y+z\right)}{2}$$

** expression

$$-\frac{\sqrt{5}Q_xz\left(3x^2-z^2\right)}{4}-\frac{\sqrt{5}Q_zx\left(x^2-3z^2\right)}{4}$$

$$\frac{\sqrt{5}Q_yz\left(3y^2-z^2\right)}{4}+\frac{\sqrt{5}Q_zy\left(y^2-3z^2\right)}{4}$$

$\vec{\mathbb{Q}}_{4,1}^{(1,-1)}[q](E,2)$, $\vec{\mathbb{Q}}_{4,2}^{(1,-1)}[q](E,2)$
** symmetry

$$\frac{\sqrt{5}yz\left(6x^2-y^2-z^2\right)}{2}$$

$$-\frac{\sqrt{5}xz\left(x^2-6y^2+z^2\right)}{2}$$

** expression

$$\frac{3\sqrt{35}Q_xxyz}{7}+\frac{\sqrt{35}Q_yz\left(6x^2-3y^2-z^2\right)}{28}+\frac{\sqrt{35}Q_zy\left(6x^2-y^2-3z^2\right)}{28}$$

$$-\frac{\sqrt{35}Q_xz\left(3x^2-6y^2+z^2\right)}{28}+\frac{3\sqrt{35}Q_yxyz}{7}-\frac{\sqrt{35}Q_zx\left(x^2-6y^2+3z^2\right)}{28}$$

$\vec{\mathbb{Q}}_{4,1}^{(1,1)}[q](E,1)$, $\vec{\mathbb{Q}}_{4,2}^{(1,1)}[q](E,1)$
** symmetry

$$-\frac{\sqrt{35}xz\left(x-z\right)\left(x+z\right)}{2}$$

$$\frac{\sqrt{35}yz\left(y-z\right)\left(y+z\right)}{2}$$

** expression

$$-\frac{\sqrt{77}Q_xz\left(6x^4-3x^2y^2-11x^2z^2+y^2z^2+z^4\right)}{22}-\frac{9\sqrt{77}Q_yxyz\left(x-z\right)\left(x+z\right)}{22}+\frac{\sqrt{77}Q_zx\left(x^4+x^2y^2-11x^2z^2-3y^2z^2+6z^4\right)}{22}$$

$$\frac{9\sqrt{77}Q_xxyz\left(y-z\right)\left(y+z\right)}{22}-\frac{\sqrt{77}Q_yz\left(3x^2y^2-x^2z^2-6y^4+11y^2z^2-z^4\right)}{22}-\frac{\sqrt{77}Q_zy\left(x^2y^2-3x^2z^2+y^4-11y^2z^2+6z^4\right)}{22}$$

$\vec{\mathbb{Q}}_{4,1}^{(1,1)}[q](E,2)$, $\vec{\mathbb{Q}}_{4,2}^{(1,1)}[q](E,2)$
** symmetry

$$\frac{\sqrt{5}yz\left(6x^2-y^2-z^2\right)}{2}$$

$$-\frac{\sqrt{5}xz\left(x^2-6y^2+z^2\right)}{2}$$

** expression

$$\frac{21\sqrt{11}Q_xxyz\left(2x^2-y^2-z^2\right)}{22}-\frac{\sqrt{11}Q_yz\left(6x^4-51x^2y^2+5x^2z^2+6y^4+5y^2z^2-z^4\right)}{22}-\frac{\sqrt{11}Q_zy\left(6x^4+5x^2y^2-51x^2z^2-y^4+5y^2z^2+6z^4\right)}{22}$$

$$-\frac{\sqrt{11}Q_xz\left(6x^4-51x^2y^2+5x^2z^2+6y^4+5y^2z^2-z^4\right)}{22}-\frac{21\sqrt{11}Q_yxyz\left(x^2-2y^2+z^2\right)}{22}+\frac{\sqrt{11}Q_zx\left(x^4-5x^2y^2-5x^2z^2-6y^4+51y^2z^2-6z^4\right)}{22}$$