PG No. 19 C_{3v} $3m$ (3m1 setting) [trigonal] (polar, internal polar quadrupole)

* Harmonics for rank 0

$\vec{\mathbb{Q}}_0^{(2,2)}[q](A_1)$

** symmetry

1

** expression

$$-\frac{\sqrt{5}Q_u\left(x^2+y^2-2z^2\right)}{10}+\frac{\sqrt{15}Q_v\left(x-y\right)\left(x+y\right)}{10}+\frac{\sqrt{15}Q_{xy}xy}{5}+\frac{\sqrt{15}Q_{xz}xz}{5}+\frac{\sqrt{15}Q_{yz}yz}{5}$$

* Harmonics for rank 1

$\vec{\mathbb{Q}}_1^{(2,0)}[q](A_1)$

** symmetry

z

** expression

$$\frac{\sqrt{10}Q_uz}{5}+\frac{\sqrt{30}Q_{xz}x}{10}+\frac{\sqrt{30}Q_{yz}y}{10}$$

$\vec{\mathbb{Q}}_1^{(2,2)}[q](A_1)$

** symmetry

z

** expression

$$-\frac{3\sqrt{35}Q_uz\left(3x^2+3y^2-2z^2\right)}{70}+\frac{\sqrt{105}Q_vz\left(x-y\right)\left(x+y\right)}{14}+\frac{\sqrt{105}Q_{xy}xyz}{7}-\frac{\sqrt{105}Q_{xz}x\left(x^2+y^2-4z^2\right)}{35}-\frac{\sqrt{105}Q_{yz}y\left(x^2+y^2-4z^2\right)}{35}$$

$\vec{\mathbb{Q}}_{1,1}^{(2,0)}[q](E)$, $\vec{\mathbb{Q}}_{1,2}^{(2,0)}[q](E)$

** symmetry

x

y

** expression

$$-\frac{\sqrt{10}Q_ux}{10}+\frac{\sqrt{30}Q_vx}{10}+\frac{\sqrt{30}Q_{xy}y}{10}+\frac{\sqrt{30}Q_{xz}z}{10}$$

$$-\frac{\sqrt{10}Q_uy}{10}-\frac{\sqrt{30}Q_vy}{10}+\frac{\sqrt{30}Q_{xy}x}{10}+\frac{\sqrt{30}Q_{yz}z}{10}$$

$\vec{\mathbb{Q}}_{1,1}^{(2,2)}[q](E)$, $\vec{\mathbb{Q}}_{1,2}^{(2,2)}[q](E)$

** symmetry

x

y

** expression

$$-\frac{3\sqrt{35}Q_ux\left(x^2+y^2-4z^2\right)}{70}+\frac{\sqrt{105}Q_vx\left(3x^2-7y^2-2z^2\right)}{70}+\frac{\sqrt{105}Q_{xy}y\left(4x^2-y^2-z^2\right)}{35}+\frac{\sqrt{105}Q_{xz}z\left(4x^2-y^2-z^2\right)}{35}+\frac{\sqrt{105}Q_{yz}xyz}{7}$$

$$-\frac{3\sqrt{35}Q_uy\left(x^2+y^2-4z^2\right)}{70}+\frac{\sqrt{105}Q_vy\left(7x^2-3y^2+2z^2\right)}{70}-\frac{\sqrt{105}Q_{xy}x\left(x^2-4y^2+z^2\right)}{35}+\frac{\sqrt{105}Q_{xz}xyz}{7}-\frac{\sqrt{105}Q_{yz}z\left(x^2-4y^2+z^2\right)}{35}$$

* Harmonics for rank 2

$\vec{\mathbb{Q}}_2^{(2,-2)}[q](A_1)$

** symmetry

$$-\frac{x^2}{2}-\frac{y^2}{2}+z^2$$

** expression

Q_u

$\vec{\mathbb{Q}}_2^{(2,0)}[q](A_1)$

** symmetry

$$-\frac{x^2}{2}-\frac{y^2}{2}+z^2$$

** expression

$$-\frac{\sqrt{14}Q_u\left(x^2+y^2-2z^2\right)}{14}-\frac{\sqrt{42}Q_v\left(x-y\right)\left(x+y\right)}{14}-\frac{\sqrt{42}Q_{xy}xy}{7}+\frac{\sqrt{42}Q_{xz}xz}{14}+\frac{\sqrt{42}Q_{yz}yz}{14}$$

$\vec{\mathbb{Q}}_2^{(2,2)}[q](A_1)$
** symmetry

$$-\frac{x^2}{2}-\frac{y^2}{2}+z^2$$

** expression

$$\frac{\sqrt{14}Q_u\left(3x^4+6x^2y^2-24x^2z^2+3y^4-24y^2z^2+8z^4\right)}{56}-\frac{5\sqrt{42}Q_v\left(x-y\right)\left(x+y\right)\left(x^2+y^2-6z^2\right)}{168}$$
$$-\frac{5\sqrt{42}Q_{xy}xy\left(x^2+y^2-6z^2\right)}{84}-\frac{5\sqrt{42}Q_{xz}xz\left(3x^2+3y^2-4z^2\right)}{84}-\frac{5\sqrt{42}Q_{yz}yz\left(3x^2+3y^2-4z^2\right)}{84}$$

$\vec{\mathbb{Q}}_{2,1}^{(2,-2)}[q](E,1)$, $\vec{\mathbb{Q}}_{2,2}^{(2,-2)}[q](E,1)$
** symmetry

$$\sqrt{3}xz$$

$$\sqrt{3}yz$$

** expression

$$Q_{xz}$$

$$Q_{yz}$$

$\vec{\mathbb{Q}}_{2,1}^{(2,-2)}[q](E,2)$, $\vec{\mathbb{Q}}_{2,2}^{(2,-2)}[q](E,2)$
** symmetry

$$\sqrt{3}xy$$

$$\frac{\sqrt{3}\left(x-y\right)\left(x+y\right)}{2}$$

** expression

$$Q_{xy}$$

$$Q_v$$

$\vec{\mathbb{Q}}_{2,1}^{(2,0)}[q](E,1)$, $\vec{\mathbb{Q}}_{2,2}^{(2,0)}[q](E,1)$
** symmetry

$$\sqrt{3}xz$$

$$\sqrt{3}yz$$

** expression

$$\frac{\sqrt{42}Q_uxz}{14}+\frac{3\sqrt{14}Q_vxz}{14}+\frac{3\sqrt{14}Q_{xy}yz}{14}+\frac{\sqrt{14}Q_{xz}\left(x^2-2y^2+z^2\right)}{14}+\frac{3\sqrt{14}Q_{yz}xy}{14}$$

$$\frac{\sqrt{42}Q_uyz}{14}-\frac{3\sqrt{14}Q_vyz}{14}+\frac{3\sqrt{14}Q_{xy}xz}{14}+\frac{3\sqrt{14}Q_{xz}xy}{14}-\frac{\sqrt{14}Q_{yz}\left(2x^2-y^2-z^2\right)}{14}$$

$\vec{\mathbb{Q}}_{2,1}^{(2,0)}[q](E,2)$, $\vec{\mathbb{Q}}_{2,2}^{(2,0)}[q](E,2)$
** symmetry

$$\sqrt{3}xy$$

$$\frac{\sqrt{3}\left(x-y\right)\left(x+y\right)}{2}$$

** expression

$$-\frac{\sqrt{42}Q_uxy}{7}+\frac{\sqrt{14}Q_{xy}\left(x^2+y^2-2z^2\right)}{14}+\frac{3\sqrt{14}Q_{xz}yz}{14}+\frac{3\sqrt{14}Q_{yz}xz}{14}$$

$$-\frac{\sqrt{42}Q_u\left(x-y\right)\left(x+y\right)}{14}+\frac{\sqrt{14}Q_v\left(x^2+y^2-2z^2\right)}{14}+\frac{3\sqrt{14}Q_{xz}xz}{14}-\frac{3\sqrt{14}Q_{yz}yz}{14}$$

$\vec{\mathbb{Q}}_{2,1}^{(2,2)}[q](E,1)$, $\vec{\mathbb{Q}}_{2,2}^{(2,2)}[q](E,1)$

** symmetry

$$\sqrt{3}xz$$

$$\sqrt{3}yz$$

** expression

$$-\frac{5\sqrt{42}Q_u xz\left(3x^2+3y^2-4z^2\right)}{84}+\frac{5\sqrt{14}Q_v xz\left(5x^2-9y^2-2z^2\right)}{84}+\frac{5\sqrt{14}Q_{xy}yz\left(6x^2-y^2-z^2\right)}{42}$$
$$-\frac{\sqrt{14}Q_{xz}\left(4x^4+3x^2y^2-27x^2z^2-y^4+3y^2z^2+4z^4\right)}{42}-\frac{5\sqrt{14}Q_{yz}xy\left(x^2+y^2-6z^2\right)}{42}$$

$$-\frac{5\sqrt{42}Q_u yz\left(3x^2+3y^2-4z^2\right)}{84}+\frac{5\sqrt{14}Q_v yz\left(9x^2-5y^2+2z^2\right)}{84}-\frac{5\sqrt{14}Q_{xy}xz\left(x^2-6y^2+z^2\right)}{42}$$
$$-\frac{5\sqrt{14}Q_{xz}xy\left(x^2+y^2-6z^2\right)}{42}+\frac{\sqrt{14}Q_{yz}\left(x^4-3x^2y^2-3x^2z^2-4y^4+27y^2z^2-4z^4\right)}{42}$$

$\vec{\mathbb{Q}}_{2,1}^{(2,2)}[q](E,2)$, $\vec{\mathbb{Q}}_{2,2}^{(2,2)}[q](E,2)$

** symmetry

$$\sqrt{3}xy$$

$$\frac{\sqrt{3}\,(x-y)\,(x+y)}{2}$$

** expression

$$-\frac{5\sqrt{42}Q_u xy\left(x^2+y^2-6z^2\right)}{84}+\frac{5\sqrt{14}Q_v xy\,(x-y)\,(x+y)}{12}-\frac{\sqrt{14}Q_{xy}\left(4x^4-27x^2y^2+3x^2z^2+4y^4+3y^2z^2-z^4\right)}{42}$$
$$+\frac{5\sqrt{14}Q_{xz}yz\left(6x^2-y^2-z^2\right)}{42}-\frac{5\sqrt{14}Q_{yz}xz\left(x^2-6y^2+z^2\right)}{42}$$

$$-\frac{5\sqrt{42}Q_u\,(x-y)\,(x+y)\left(x^2+y^2-6z^2\right)}{168}+\frac{\sqrt{14}Q_v\left(19x^4-102x^2y^2-12x^2z^2+19y^4-12y^2z^2+4z^4\right)}{168}$$
$$+\frac{5\sqrt{14}Q_{xy}xy\,(x-y)\,(x+y)}{12}+\frac{5\sqrt{14}Q_{xz}xz\left(5x^2-9y^2-2z^2\right)}{84}+\frac{5\sqrt{14}Q_{yz}yz\left(9x^2-5y^2+2z^2\right)}{84}$$

* Harmonics for rank 3

$\vec{\mathbb{Q}}_3^{(2,-2)}[q](A_1,1)$

** symmetry

$$-\frac{z\left(3x^2+3y^2-2z^2\right)}{2}$$

** expression

$$\frac{\sqrt{15}Q_u z}{5}-\frac{\sqrt{5}Q_{xz}x}{5}-\frac{\sqrt{5}Q_{yz}y}{5}$$

$\vec{\mathbb{Q}}_3^{(2,-2)}[q](A_1,2)$

** symmetry

$$\frac{\sqrt{10}y\left(3x^2-y^2\right)}{4}$$

** expression

$$\frac{\sqrt{2}Q_v y}{2}+\frac{\sqrt{2}Q_{xy}x}{2}$$

$\vec{\mathbb{Q}}_3^{(2,0)}[q](A_1,1)$

** symmetry

$$-\frac{z\left(3x^2+3y^2-2z^2\right)}{2}$$

** expression

$$-\frac{\sqrt{15}Q_u z\left(3x^2+3y^2-2z^2\right)}{15}-\frac{\sqrt{5}Q_v z\,(x-y)\,(x+y)}{2}-\sqrt{5}Q_{xy}xyz-\frac{\sqrt{5}Q_{xz}x\left(x^2+y^2-4z^2\right)}{20}-\frac{\sqrt{5}Q_{yz}y\left(x^2+y^2-4z^2\right)}{20}$$

$\vec{\mathbb{Q}}_3^{(2,0)}[q](A_1,2)$

** symmetry

$$\frac{\sqrt{10}y\left(3x^2-y^2\right)}{4}$$

** expression

$$-\frac{5\sqrt{6}Q_u y\left(3x^2-y^2\right)}{24}+\frac{\sqrt{2}Q_v y\left(x^2+y^2-4z^2\right)}{8}+\frac{\sqrt{2}Q_{xy}x\left(x^2+y^2-4z^2\right)}{8}+\frac{5\sqrt{2}Q_{xz}xyz}{4}+\frac{5\sqrt{2}Q_{yz}z\left(x-y\right)\left(x+y\right)}{8}$$

$\vec{\mathbb{Q}}_3^{(2,2)}[q](A_1,1)$
** symmetry

$$-\frac{z\left(3x^2+3y^2-2z^2\right)}{2}$$

** expression

$$\frac{\sqrt{330}Q_u z\left(15x^4+30x^2y^2-40x^2z^2+15y^4-40y^2z^2+8z^4\right)}{264}-\frac{7\sqrt{110}Q_v z\left(x-y\right)\left(x+y\right)\left(x^2+y^2-2z^2\right)}{88}-\frac{7\sqrt{110}Q_{xy}xyz\left(x^2+y^2-2z^2\right)}{44}$$
$$+\frac{\sqrt{110}Q_{xz}x\left(x^4+2x^2y^2-12x^2z^2+y^4-12y^2z^2+8z^4\right)}{44}+\frac{\sqrt{110}Q_{yz}y\left(x^4+2x^2y^2-12x^2z^2+y^4-12y^2z^2+8z^4\right)}{44}$$

$\vec{\mathbb{Q}}_3^{(2,2)}[q](A_1,2)$
** symmetry

$$\frac{\sqrt{10}y\left(3x^2-y^2\right)}{4}$$

** expression

$$-\frac{7\sqrt{33}Q_u y\left(3x^2-y^2\right)\left(x^2+y^2-8z^2\right)}{264}+\frac{\sqrt{11}Q_v y\left(53x^4-104x^2y^2-6x^2z^2+11y^4-6y^2z^2+4z^4\right)}{88}$$
$$-\frac{\sqrt{11}Q_{xy}x\left(5x^4-53x^2y^2+3x^2z^2+26y^4+3y^2z^2-2z^4\right)}{44}+\frac{7\sqrt{11}Q_{xz}xyz\left(7x^2-5y^2-2z^2\right)}{44}-\frac{7\sqrt{11}Q_{yz}z\left(x^4-9x^2y^2+x^2z^2+2y^4-y^2z^2\right)}{44}$$

$\vec{\mathbb{Q}}_3^{(2,-2)}[q](A_2)$
** symmetry

$$\frac{\sqrt{10}x\left(x^2-3y^2\right)}{4}$$

** expression

$$\frac{\sqrt{2}Q_v x}{2}-\frac{\sqrt{2}Q_{xy}y}{2}$$

$\vec{\mathbb{Q}}_3^{(2,0)}[q](A_2)$
** symmetry

$$\frac{\sqrt{10}x\left(x^2-3y^2\right)}{4}$$

** expression

$$-\frac{5\sqrt{6}Q_u x\left(x^2-3y^2\right)}{24}+\frac{\sqrt{2}Q_v x\left(x^2+y^2-4z^2\right)}{8}-\frac{\sqrt{2}Q_{xy}y\left(x^2+y^2-4z^2\right)}{8}+\frac{5\sqrt{2}Q_{xz}z\left(x-y\right)\left(x+y\right)}{8}-\frac{5\sqrt{2}Q_{yz}xyz}{4}$$

$\vec{\mathbb{Q}}_3^{(2,2)}[q](A_2)$
** symmetry

$$\frac{\sqrt{10}x\left(x^2-3y^2\right)}{4}$$

** expression

$$-\frac{7\sqrt{33}Q_u x\left(x^2-3y^2\right)\left(x^2+y^2-8z^2\right)}{264}+\frac{\sqrt{11}Q_v x\left(11x^4-104x^2y^2-6x^2z^2+53y^4-6y^2z^2+4z^4\right)}{88}$$
$$+\frac{\sqrt{11}Q_{xy}y\left(26x^4-53x^2y^2+3x^2z^2+5y^4+3y^2z^2-2z^4\right)}{44}+\frac{7\sqrt{11}Q_{xz}z\left(2x^4-9x^2y^2-x^2z^2+y^4+y^2z^2\right)}{44}+\frac{7\sqrt{11}Q_{yz}xyz\left(5x^2-7y^2+2z^2\right)}{44}$$

$\vec{\mathbb{Q}}_{3,1}^{(2,-2)}[q](E,1)$, $\vec{\mathbb{Q}}_{3,2}^{(2,-2)}[q](E,1)$
** symmetry

$$-\frac{\sqrt{6}x\left(x^2+y^2-4z^2\right)}{4}$$

$$-\frac{\sqrt{6}y\left(x^2+y^2-4z^2\right)}{4}$$

** expression

$$\frac{\sqrt{10}Q_u x}{5}-\frac{\sqrt{30}Q_v x}{30}-\frac{\sqrt{30}Q_{xy}y}{30}+\frac{2\sqrt{30}Q_{xz}z}{15}$$

$$\frac{\sqrt{10}Q_u y}{5}+\frac{\sqrt{30}Q_v y}{30}-\frac{\sqrt{30}Q_{xy}x}{30}+\frac{2\sqrt{30}Q_{yz}z}{15}$$

$\vec{\mathbb{Q}}_{3,1}^{(2,-2)}[q](E,2)$, $\vec{\mathbb{Q}}_{3,2}^{(2,-2)}[q](E,2)$

** symmetry

$$\sqrt{15}xyz$$

$$\frac{\sqrt{15}z\left(x-y\right)\left(x+y\right)}{2}$$

** expression

$$\frac{\sqrt{3}Q_{xy}z}{3}+\frac{\sqrt{3}Q_{xz}y}{3}+\frac{\sqrt{3}Q_{yz}x}{3}$$

$$\frac{\sqrt{3}Q_v z}{3}+\frac{\sqrt{3}Q_{xz}x}{3}-\frac{\sqrt{3}Q_{yz}y}{3}$$

$\vec{\mathbb{Q}}_{3,1}^{(2,0)}[q](E,1)$, $\vec{\mathbb{Q}}_{3,2}^{(2,0)}[q](E,1)$

** symmetry

$$-\frac{\sqrt{6}x\left(x^2+y^2-4z^2\right)}{4}$$

$$-\frac{\sqrt{6}y\left(x^2+y^2-4z^2\right)}{4}$$

** expression

$$-\frac{3\sqrt{10}Q_u x\left(x^2+y^2-4z^2\right)}{40}-\frac{\sqrt{30}Q_v x\left(11x^2-9y^2-24z^2\right)}{120}-\frac{\sqrt{30}Q_{xy}y\left(21x^2+y^2-24z^2\right)}{120}+\frac{\sqrt{30}Q_{xz}z\left(9x^2-21y^2+4z^2\right)}{120}+\frac{\sqrt{30}Q_{yz}xyz}{4}$$

$$-\frac{3\sqrt{10}Q_u y\left(x^2+y^2-4z^2\right)}{40}-\frac{\sqrt{30}Q_v y\left(9x^2-11y^2+24z^2\right)}{120}-\frac{\sqrt{30}Q_{xy}x\left(x^2+21y^2-24z^2\right)}{120}+\frac{\sqrt{30}Q_{xz}xyz}{4}-\frac{\sqrt{30}Q_{yz}z\left(21x^2-9y^2-4z^2\right)}{120}$$

$\vec{\mathbb{Q}}_{3,1}^{(2,0)}[q](E,2)$, $\vec{\mathbb{Q}}_{3,2}^{(2,0)}[q](E,2)$

** symmetry

$$\sqrt{15}xyz$$

$$\frac{\sqrt{15}z\left(x-y\right)\left(x+y\right)}{2}$$

** expression

$$\frac{\sqrt{3}Q_{xy}z\left(3x^2+3y^2-2z^2\right)}{6}+\frac{\sqrt{3}Q_{xz}y\left(3x^2-2y^2+3z^2\right)}{6}-\frac{\sqrt{3}Q_{yz}x\left(2x^2-3y^2-3z^2\right)}{6}$$

$$\frac{\sqrt{3}Q_v z\left(3x^2+3y^2-2z^2\right)}{6}+\frac{\sqrt{3}Q_{xz}x\left(x^2-9y^2+6z^2\right)}{12}+\frac{\sqrt{3}Q_{yz}y\left(9x^2-y^2-6z^2\right)}{12}$$

$\vec{\mathbb{Q}}_{3,1}^{(2,2)}[q](E,1)$, $\vec{\mathbb{Q}}_{3,2}^{(2,2)}[q](E,1)$

** symmetry

$$-\frac{\sqrt{6}x\left(x^2+y^2-4z^2\right)}{4}$$

$$-\frac{\sqrt{6}y\left(x^2+y^2-4z^2\right)}{4}$$

** expression

$$\frac{3\sqrt{55}Q_u x\left(x^4+2x^2y^2-12x^2z^2+y^4-12y^2z^2+8z^4\right)}{88}-\frac{\sqrt{165}Q_v x\left(5x^4-4x^2y^2-46x^2z^2-9y^4+66y^2z^2+12z^4\right)}{264}$$
$$-\frac{\sqrt{165}Q_{xy}y\left(6x^4+5x^2y^2-51x^2z^2-y^4+5y^2z^2+6z^4\right)}{132}$$
$$-\frac{\sqrt{165}Q_{xz}z\left(18x^4+15x^2y^2-41x^2z^2-3y^4+y^2z^2+4z^4\right)}{132}-\frac{7\sqrt{165}Q_{yz}xyz\left(x^2+y^2-2z^2\right)}{44}$$

$$\frac{3\sqrt{55}Q_u y\left(x^4+2x^2y^2-12x^2z^2+y^4-12y^2z^2+8z^4\right)}{88}-\frac{\sqrt{165}Q_v y\left(9x^4+4x^2y^2-66x^2z^2-5y^4+46y^2z^2-12z^4\right)}{264}$$
$$+\frac{\sqrt{165}Q_{xy}x\left(x^4-5x^2y^2-5x^2z^2-6y^4+51y^2z^2-6z^4\right)}{132}-\frac{7\sqrt{165}Q_{xz}xyz\left(x^2+y^2-2z^2\right)}{44}$$
$$+\frac{\sqrt{165}Q_{yz}z\left(3x^4-15x^2y^2-x^2z^2-18y^4+41y^2z^2-4z^4\right)}{132}$$

$\vec{\mathbb{Q}}_{3,1}^{(2,2)}[q](E,2)$, $\vec{\mathbb{Q}}_{3,2}^{(2,2)}[q](E,2)$

** symmetry

$$\sqrt{15}xyz$$

$$\frac{\sqrt{15}z\left(x-y\right)\left(x+y\right)}{2}$$

** expression

$$-\frac{21\sqrt{22}Q_{u}xyz\left(x^{2}+y^{2}-2z^{2}\right)}{44}+\frac{21\sqrt{66}Q_{v}xyz\left(x-y\right)\left(x+y\right)}{44}-\frac{\sqrt{66}Q_{xy}z\left(6x^{4}-51x^{2}y^{2}+5x^{2}z^{2}+6y^{4}+5y^{2}z^{2}-z^{4}\right)}{66}$$
$$-\frac{\sqrt{66}Q_{xz}y\left(6x^{4}+5x^{2}y^{2}-51x^{2}z^{2}-y^{4}+5y^{2}z^{2}+6z^{4}\right)}{66}+\frac{\sqrt{66}Q_{yz}x\left(x^{4}-5x^{2}y^{2}-5x^{2}z^{2}-6y^{4}+51y^{2}z^{2}-6z^{4}\right)}{66}$$

$$-\frac{21\sqrt{22}Q_{u}z\left(x-y\right)\left(x+y\right)\left(x^{2}+y^{2}-2z^{2}\right)}{88}+\frac{\sqrt{66}Q_{v}z\left(39x^{4}-174x^{2}y^{2}-20x^{2}z^{2}+39y^{4}-20y^{2}z^{2}+4z^{4}\right)}{264}+\frac{21\sqrt{66}Q_{xy}xyz\left(x-y\right)\left(x+y\right)}{44}$$
$$-\frac{\sqrt{66}Q_{xz}x\left(5x^{4}-4x^{2}y^{2}-46x^{2}z^{2}-9y^{4}+66y^{2}z^{2}+12z^{4}\right)}{132}-\frac{\sqrt{66}Q_{yz}y\left(9x^{4}+4x^{2}y^{2}-66x^{2}z^{2}-5y^{4}+46y^{2}z^{2}-12z^{4}\right)}{132}$$

* Harmonics for rank 4

$\vec{\mathbb{Q}}_{4}^{(2,-2)}[q](A_1,1)$

** symmetry

$$\frac{3x^{4}}{8}+\frac{3x^{2}y^{2}}{4}-3x^{2}z^{2}+\frac{3y^{4}}{8}-3y^{2}z^{2}+z^{4}$$

** expression

$$-\frac{3\sqrt{70}Q_{u}\left(x^{2}+y^{2}-2z^{2}\right)}{70}+\frac{\sqrt{210}Q_{v}\left(x-y\right)\left(x+y\right)}{140}+\frac{\sqrt{210}Q_{xy}xy}{70}-\frac{2\sqrt{210}Q_{xz}xz}{35}-\frac{2\sqrt{210}Q_{yz}yz}{35}$$

$\vec{\mathbb{Q}}_{4}^{(2,-2)}[q](A_1,2)$

** symmetry

$$\frac{\sqrt{70}yz\left(3x^{2}-y^{2}\right)}{4}$$

** expression

$$\frac{\sqrt{3}Q_{v}yz}{2}+\frac{\sqrt{3}Q_{xy}xz}{2}+\frac{\sqrt{3}Q_{xz}xy}{2}+\frac{\sqrt{3}Q_{yz}\left(x-y\right)\left(x+y\right)}{4}$$

$\vec{\mathbb{Q}}_{4}^{(2,0)}[q](A_1,1)$

** symmetry

$$\frac{3x^{4}}{8}+\frac{3x^{2}y^{2}}{4}-3x^{2}z^{2}+\frac{3y^{4}}{8}-3y^{2}z^{2}+z^{4}$$

** expression

$$\frac{\sqrt{385}Q_{u}\left(3x^{4}+6x^{2}y^{2}-24x^{2}z^{2}+3y^{4}-24y^{2}z^{2}+8z^{4}\right)}{308}+\frac{3\sqrt{1155}Q_{v}\left(x-y\right)\left(x+y\right)\left(x^{2}+y^{2}-6z^{2}\right)}{308}$$
$$+\frac{3\sqrt{1155}Q_{xy}xy\left(x^{2}+y^{2}-6z^{2}\right)}{154}-\frac{\sqrt{1155}Q_{xz}xz\left(3x^{2}+3y^{2}-4z^{2}\right)}{308}-\frac{\sqrt{1155}Q_{yz}yz\left(3x^{2}+3y^{2}-4z^{2}\right)}{308}$$

$\vec{\mathbb{Q}}_{4}^{(2,0)}[q](A_1,2)$

** symmetry

$$\frac{\sqrt{70}yz\left(3x^{2}-y^{2}\right)}{4}$$

** expression

$$-\frac{7\sqrt{22}Q_{u}yz\left(3x^{2}-y^{2}\right)}{88}+\frac{3\sqrt{66}Q_{v}yz\left(3x^{2}+3y^{2}-4z^{2}\right)}{88}+\frac{3\sqrt{66}Q_{xy}xz\left(3x^{2}+3y^{2}-4z^{2}\right)}{88}$$
$$+\frac{\sqrt{66}Q_{xz}xy\left(9x^{2}-19y^{2}+30z^{2}\right)}{88}-\frac{\sqrt{66}Q_{yz}\left(6x^{4}-21x^{2}y^{2}-15x^{2}z^{2}+y^{4}+15y^{2}z^{2}\right)}{88}$$

$\vec{\mathbb{Q}}_{4}^{(2,2)}[q](A_1,1)$

** symmetry

$$\frac{3x^{4}}{8}+\frac{3x^{2}y^{2}}{4}-3x^{2}z^{2}+\frac{3y^{4}}{8}-3y^{2}z^{2}+z^{4}$$

** expression

$$-\frac{3\sqrt{715}Q_{u}\left(5x^{6}+15x^{4}y^{2}-90x^{4}z^{2}+15x^{2}y^{4}-180x^{2}y^{2}z^{2}+120x^{2}z^{4}+5y^{6}-90y^{4}z^{2}+120y^{2}z^{4}-16z^{6}\right)}{2288}$$
$$+\frac{7\sqrt{2145}Q_{v}\left(x-y\right)\left(x+y\right)\left(x^{4}+2x^{2}y^{2}-16x^{2}z^{2}+y^{4}-16y^{2}z^{2}+16z^{4}\right)}{2288}+\frac{7\sqrt{2145}Q_{xy}xy\left(x^{4}+2x^{2}y^{2}-16x^{2}z^{2}+y^{4}-16y^{2}z^{2}+16z^{4}\right)}{1144}$$
$$+\frac{7\sqrt{2145}Q_{xz}xz\left(5x^{4}+10x^{2}y^{2}-20x^{2}z^{2}+5y^{4}-20y^{2}z^{2}+8z^{4}\right)}{1144}+\frac{7\sqrt{2145}Q_{yz}yz\left(5x^{4}+10x^{2}y^{2}-20x^{2}z^{2}+5y^{4}-20y^{2}z^{2}+8z^{4}\right)}{1144}$$

$\vec{\mathbb{Q}}_4^{(2,2)}[q](A_1,2)$

** symmetry

$$\frac{\sqrt{70}yz\left(3x^2-y^2\right)}{4}$$

** expression

$$-\frac{9\sqrt{2002}Q_uyz\left(3x^2-y^2\right)\left(3x^2+3y^2-8z^2\right)}{1144}+\frac{\sqrt{6006}Q_vyz\left(85x^4-160x^2y^2-10x^2z^2+19y^4-10y^2z^2+4z^4\right)}{1144}$$
$$-\frac{\sqrt{6006}Q_{xy}xz\left(7x^4-85x^2y^2+5x^2z^2+40y^4+5y^2z^2-2z^4\right)}{572}-\frac{\sqrt{6006}Q_{xz}xy\left(7x^4+2x^2y^2-76x^2z^2-5y^4+44y^2z^2+16z^4\right)}{572}$$
$$+\frac{\sqrt{6006}Q_{yz}\left(x^6-8x^4y^2-7x^4z^2-7x^2y^4+90x^2y^2z^2-8x^2z^4+2y^6-23y^4z^2+8y^2z^4\right)}{572}$$

$\vec{\mathbb{Q}}_4^{(2,-2)}[q](A_2)$

** symmetry

$$\frac{\sqrt{70}xz\left(x^2-3y^2\right)}{4}$$

** expression

$$\frac{\sqrt{3}Q_vxz}{2}-\frac{\sqrt{3}Q_{xy}yz}{2}+\frac{\sqrt{3}Q_{xz}\left(x-y\right)\left(x+y\right)}{4}-\frac{\sqrt{3}Q_{yz}xy}{2}$$

$\vec{\mathbb{Q}}_4^{(2,0)}[q](A_2)$

** symmetry

$$\frac{\sqrt{70}xz\left(x^2-3y^2\right)}{4}$$

** expression

$$-\frac{7\sqrt{22}Q_uxz\left(x^2-3y^2\right)}{88}+\frac{3\sqrt{66}Q_vxz\left(3x^2+3y^2-4z^2\right)}{88}-\frac{3\sqrt{66}Q_{xy}yz\left(3x^2+3y^2-4z^2\right)}{88}$$
$$+\frac{\sqrt{66}Q_{xz}\left(x^4-21x^2y^2+15x^2z^2+6y^4-15y^2z^2\right)}{88}+\frac{\sqrt{66}Q_{yz}xy\left(19x^2-9y^2-30z^2\right)}{88}$$

$\vec{\mathbb{Q}}_4^{(2,2)}[q](A_2)$

** symmetry

$$\frac{\sqrt{70}xz\left(x^2-3y^2\right)}{4}$$

** expression

$$-\frac{9\sqrt{2002}Q_uxz\left(x^2-3y^2\right)\left(3x^2+3y^2-8z^2\right)}{1144}+\frac{\sqrt{6006}Q_vxz\left(19x^4-160x^2y^2-10x^2z^2+85y^4-10y^2z^2+4z^4\right)}{1144}$$
$$+\frac{\sqrt{6006}Q_{xy}yz\left(40x^4-85x^2y^2+5x^2z^2+7y^4+5y^2z^2-2z^4\right)}{572}$$
$$-\frac{\sqrt{6006}Q_{xz}\left(2x^6-7x^4y^2-23x^4z^2-8x^2y^4+90x^2y^2z^2+8x^2z^4+y^6-7y^4z^2-8y^2z^4\right)}{572}$$
$$-\frac{\sqrt{6006}Q_{yz}xy\left(5x^4-2x^2y^2-44x^2z^2-7y^4+76y^2z^2-16z^4\right)}{572}$$

$\vec{\mathbb{Q}}_{4,1}^{(2,-2)}[q](E,1)$, $\vec{\mathbb{Q}}_{4,2}^{(2,-2)}[q](E,1)$

** symmetry

$$-\frac{\sqrt{10}xz\left(3x^2+3y^2-4z^2\right)}{4}$$

$$-\frac{\sqrt{10}yz\left(3x^2+3y^2-4z^2\right)}{4}$$

** expression

$$\frac{3\sqrt{7}Q_uxz}{7}-\frac{\sqrt{21}Q_vxz}{14}-\frac{\sqrt{21}Q_{xy}yz}{14}-\frac{\sqrt{21}Q_{xz}\left(3x^2+y^2-4z^2\right)}{28}-\frac{\sqrt{21}Q_{yz}xy}{14}$$

$$\frac{3\sqrt{7}Q_uyz}{7}+\frac{\sqrt{21}Q_vyz}{14}-\frac{\sqrt{21}Q_{xy}xz}{14}-\frac{\sqrt{21}Q_{xz}xy}{14}-\frac{\sqrt{21}Q_{yz}\left(x^2+3y^2-4z^2\right)}{28}$$

$\vec{\mathbb{Q}}_{4,1}^{(2,-2)}[q](E,2)$, $\vec{\mathbb{Q}}_{4,2}^{(2,-2)}[q](E,2)$

** symmetry

$$-\frac{\sqrt{35}xy\left(x-y\right)\left(x+y\right)}{2}$$

$$\frac{\sqrt{35}\left(x^2-2xy-y^2\right)\left(x^2+2xy-y^2\right)}{8}$$

** expression

$$-\frac{\sqrt{6}Q_v xy}{2}-\frac{\sqrt{6}Q_{xy}\left(x-y\right)\left(x+y\right)}{4}$$

$$\frac{\sqrt{6}Q_v\left(x-y\right)\left(x+y\right)}{4}-\frac{\sqrt{6}Q_{xy}xy}{2}$$

$\vec{\mathbb{Q}}_{4,1}^{(2,-2)}[q](E,3)$, $\vec{\mathbb{Q}}_{4,2}^{(2,-2)}[q](E,3)$

** symmetry

$$-\frac{\sqrt{5}xy\left(x^2+y^2-6z^2\right)}{2}$$

$$-\frac{\sqrt{5}\left(x-y\right)\left(x+y\right)\left(x^2+y^2-6z^2\right)}{4}$$

** expression

$$\frac{3\sqrt{14}Q_u xy}{14}-\frac{\sqrt{42}Q_{xy}\left(x^2+y^2-2z^2\right)}{28}+\frac{\sqrt{42}Q_{xz}yz}{7}+\frac{\sqrt{42}Q_{yz}xz}{7}$$

$$\frac{3\sqrt{14}Q_u\left(x-y\right)\left(x+y\right)}{28}-\frac{\sqrt{42}Q_v\left(x^2+y^2-2z^2\right)}{28}+\frac{\sqrt{42}Q_{xz}xz}{7}-\frac{\sqrt{42}Q_{yz}yz}{7}$$

$\vec{\mathbb{Q}}_{4,1}^{(2,0)}[q](E,1)$, $\vec{\mathbb{Q}}_{4,2}^{(2,0)}[q](E,1)$

** symmetry

$$-\frac{\sqrt{10}xz\left(3x^2+3y^2-4z^2\right)}{4}$$

$$-\frac{\sqrt{10}yz\left(3x^2+3y^2-4z^2\right)}{4}$$

** expression

$$-\frac{17\sqrt{154}Q_u xz\left(3x^2+3y^2-4z^2\right)}{616}-\frac{\sqrt{462}Q_v xz\left(51x^2-33y^2-40z^2\right)}{616}-\frac{\sqrt{462}Q_{xy}yz\left(93x^2+9y^2-40z^2\right)}{616}$$
$$-\frac{\sqrt{462}Q_{xz}\left(3x^4-3x^2y^2-15x^2z^2-6y^4+39y^2z^2-4z^4\right)}{616}-\frac{9\sqrt{462}Q_{yz}xy\left(x^2+y^2-6z^2\right)}{616}$$

$$-\frac{17\sqrt{154}Q_u yz\left(3x^2+3y^2-4z^2\right)}{616}-\frac{\sqrt{462}Q_v yz\left(33x^2-51y^2+40z^2\right)}{616}-\frac{\sqrt{462}Q_{xy}xz\left(9x^2+93y^2-40z^2\right)}{616}$$
$$-\frac{9\sqrt{462}Q_{xz}xy\left(x^2+y^2-6z^2\right)}{616}+\frac{\sqrt{462}Q_{yz}\left(6x^4+3x^2y^2-39x^2z^2-3y^4+15y^2z^2+4z^4\right)}{616}$$

$\vec{\mathbb{Q}}_{4,1}^{(2,0)}[q](E,2)$, $\vec{\mathbb{Q}}_{4,2}^{(2,0)}[q](E,2)$

** symmetry

$$-\frac{\sqrt{35}xy\left(x-y\right)\left(x+y\right)}{2}$$

$$\frac{\sqrt{35}\left(x^2-2xy-y^2\right)\left(x^2+2xy-y^2\right)}{8}$$

** expression

$$\frac{7\sqrt{11}Q_u xy\left(x-y\right)\left(x+y\right)}{11}-\frac{\sqrt{33}Q_v xy\left(x^2+y^2-6z^2\right)}{22}-\frac{\sqrt{33}Q_{xy}\left(x-y\right)\left(x+y\right)\left(x^2+y^2-6z^2\right)}{44}-\frac{7\sqrt{33}Q_{xz}yz\left(3x^2-y^2\right)}{44}-\frac{7\sqrt{33}Q_{yz}xz\left(x^2-3y^2\right)}{44}$$

$$-\frac{7\sqrt{11}Q_u\left(x^2-2xy-y^2\right)\left(x^2+2xy-y^2\right)}{44}+\frac{\sqrt{33}Q_v\left(x-y\right)\left(x+y\right)\left(x^2+y^2-6z^2\right)}{44}$$
$$-\frac{\sqrt{33}Q_{xy}xy\left(x^2+y^2-6z^2\right)}{22}+\frac{7\sqrt{33}Q_{xz}xz\left(x^2-3y^2\right)}{44}-\frac{7\sqrt{33}Q_{yz}yz\left(3x^2-y^2\right)}{44}$$

$\vec{\mathbb{Q}}_{4,1}^{(2,0)}[q](E,3)$, $\vec{\mathbb{Q}}_{4,2}^{(2,0)}[q](E,3)$

** symmetry

$$-\frac{\sqrt{5}xy\left(x^2+y^2-6z^2\right)}{2}$$

$$-\frac{\sqrt{5}\left(x-y\right)\left(x+y\right)\left(x^2+y^2-6z^2\right)}{4}$$

** expression

$$-\frac{2\sqrt{77}Q_u xy\left(x^2+y^2-6z^2\right)}{77}-\frac{\sqrt{231}Q_v xy\left(x-y\right)\left(x+y\right)}{22}-\frac{\sqrt{231}Q_{xy}\left(x^4+30x^2y^2-36x^2z^2+y^4-36y^2z^2+12z^4\right)}{308}$$
$$+\frac{\sqrt{231}Q_{xz}yz\left(39x^2-31y^2+18z^2\right)}{308}-\frac{\sqrt{231}Q_{yz}xz\left(31x^2-39y^2-18z^2\right)}{308}$$

$$-\frac{\sqrt{77}Q_u\left(x-y\right)\left(x+y\right)\left(x^2+y^2-6z^2\right)}{77}-\frac{\sqrt{231}Q_v\left(2x^4-3x^2y^2-9x^2z^2+2y^4-9y^2z^2+3z^4\right)}{77}$$
$$-\frac{\sqrt{231}Q_{xy}xy\left(x-y\right)\left(x+y\right)}{22}+\frac{\sqrt{231}Q_{xz}xz\left(2x^2-33y^2+9z^2\right)}{154}+\frac{\sqrt{231}Q_{yz}yz\left(33x^2-2y^2-9z^2\right)}{154}$$

$\vec{\mathbb{Q}}_{4,1}^{(2,2)}[q](E,1)$, $\vec{\mathbb{Q}}_{4,2}^{(2,2)}[q](E,1)$
** symmetry

$$-\frac{\sqrt{10}xz\left(3x^2+3y^2-4z^2\right)}{4}$$

$$-\frac{\sqrt{10}yz\left(3x^2+3y^2-4z^2\right)}{4}$$

** expression

$$\frac{21\sqrt{286}Q_u xz\left(5x^4+10x^2y^2-20x^2z^2+5y^4-20y^2z^2+8z^4\right)}{1144}$$
$$-\frac{7\sqrt{858}Q_v xz\left(7x^4-4x^2y^2-22x^2z^2-11y^4+26y^2z^2+4z^4\right)}{1144}-\frac{7\sqrt{858}Q_{xy}yz\left(8x^4+7x^2y^2-23x^2z^2-y^4+y^2z^2+2z^4\right)}{572}$$
$$+\frac{\sqrt{858}Q_{xz}\left(6x^6+11x^4y^2-101x^4z^2+4x^2y^4-90x^2y^2z^2+116x^2z^4-y^6+11y^4z^2+4y^2z^4-8z^6\right)}{572}$$
$$+\frac{7\sqrt{858}Q_{yz}xy\left(x^4+2x^2y^2-16x^2z^2+y^4-16y^2z^2+16z^4\right)}{572}$$

$$\frac{21\sqrt{286}Q_u yz\left(5x^4+10x^2y^2-20x^2z^2+5y^4-20y^2z^2+8z^4\right)}{1144}-\frac{7\sqrt{858}Q_v yz\left(11x^4+4x^2y^2-26x^2z^2-7y^4+22y^2z^2-4z^4\right)}{1144}$$
$$+\frac{7\sqrt{858}Q_{xy}xz\left(x^4-7x^2y^2-x^2z^2-8y^4+23y^2z^2-2z^4\right)}{572}+\frac{7\sqrt{858}Q_{xz}xy\left(x^4+2x^2y^2-16x^2z^2+y^4-16y^2z^2+16z^4\right)}{572}$$
$$-\frac{\sqrt{858}Q_{yz}\left(x^6-4x^4y^2-11x^4z^2-11x^2y^4+90x^2y^2z^2-4x^2z^4-6y^6+101y^4z^2-116y^2z^4+8z^6\right)}{572}$$

$\vec{\mathbb{Q}}_{4,1}^{(2,2)}[q](E,2)$, $\vec{\mathbb{Q}}_{4,2}^{(2,2)}[q](E,2)$
** symmetry

$$-\frac{\sqrt{35}xy\left(x-y\right)\left(x+y\right)}{2}$$

$$\frac{\sqrt{35}\left(x^2-2xy-y^2\right)\left(x^2+2xy-y^2\right)}{8}$$

** expression

$$\frac{9\sqrt{1001}Q_u xy\left(x-y\right)\left(x+y\right)\left(x^2+y^2-10z^2\right)}{572}-\frac{\sqrt{3003}Q_v xy\left(25x^4-82x^2y^2-4x^2z^2+25y^4-4y^2z^2+4z^4\right)}{572}$$
$$+\frac{\sqrt{3003}Q_{xy}\left(x-y\right)\left(x+y\right)\left(2x^4-29x^2y^2+x^2z^2+2y^4+y^2z^2-z^4\right)}{286}$$
$$-\frac{3\sqrt{3003}Q_{xz}yz\left(8x^4-13x^2y^2-3x^2z^2+y^4+y^2z^2\right)}{286}+\frac{3\sqrt{3003}Q_{yz}xz\left(x^4-13x^2y^2+x^2z^2+8y^4-3y^2z^2\right)}{286}$$

$$-\frac{9\sqrt{1001}Q_u\left(x^2+y^2-10z^2\right)\left(x^2-2xy-y^2\right)\left(x^2+2xy-y^2\right)}{2288}+\frac{\sqrt{3003}Q_v\left(x-y\right)\left(x+y\right)\left(17x^4-230x^2y^2-8x^2z^2+17y^4-8y^2z^2+8z^4\right)}{2288}$$
$$+\frac{\sqrt{3003}Q_{xy}xy\left(49x^4-166x^2y^2+8x^2z^2+49y^4+8y^2z^2-8z^4\right)}{1144}$$
$$+\frac{3\sqrt{3003}Q_{xz}xz\left(7x^4-58x^2y^2-4x^2z^2+23y^4+12y^2z^2\right)}{1144}+\frac{3\sqrt{3003}Q_{yz}yz\left(23x^4-58x^2y^2+12x^2z^2+7y^4-4y^2z^2\right)}{1144}$$

$\vec{\mathbb{Q}}_{4,1}^{(2,2)}[q](E,3)$, $\vec{\mathbb{Q}}_{4,2}^{(2,2)}[q](E,3)$
** symmetry

$$-\frac{\sqrt{5}xy\left(x^2+y^2-6z^2\right)}{2}$$

$$-\frac{\sqrt{5}\left(x-y\right)\left(x+y\right)\left(x^2+y^2-6z^2\right)}{4}$$

** expression

$$\frac{21\sqrt{143}Q_u xy\left(x^4+2x^2y^2-16x^2z^2+y^4-16y^2z^2+16z^4\right)}{572}-\frac{21\sqrt{429}Q_v xy\left(x-y\right)\left(x+y\right)\left(x^2+y^2-10z^2\right)}{572}$$
$$+\frac{\sqrt{429}Q_{xy}\left(2x^6-15x^4y^2-15x^4z^2-15x^2y^4+180x^2y^2z^2-15x^2z^4+2y^6-15y^4z^2-15y^2z^4+2z^6\right)}{286}$$
$$-\frac{7\sqrt{429}Q_{xz}yz\left(8x^4+7x^2y^2-23x^2z^2-y^4+y^2z^2+2z^4\right)}{286}+\frac{7\sqrt{429}Q_{yz}xz\left(x^4-7x^2y^2-x^2z^2-8y^4+23y^2z^2-2z^4\right)}{286}$$

$$\frac{21\sqrt{143}Q_u\left(x-y\right)\left(x+y\right)\left(x^4+2x^2y^2-16x^2z^2+y^4-16y^2z^2+16z^4\right)}{1144}$$
$$-\frac{\sqrt{429}Q_v\left(13x^6-45x^4y^2-150x^4z^2-45x^2y^4+540x^2y^2z^2+60x^2z^4+13y^6-150y^4z^2+60y^2z^4-8z^6\right)}{1144}$$
$$-\frac{21\sqrt{429}Q_{xy}xy\left(x-y\right)\left(x+y\right)\left(x^2+y^2-10z^2\right)}{572}-\frac{7\sqrt{429}Q_{xz}xz\left(7x^4-4x^2y^2-22x^2z^2-11y^4+26y^2z^2+4z^4\right)}{572}$$
$$-\frac{7\sqrt{429}Q_{yz}yz\left(11x^4+4x^2y^2-26x^2z^2-7y^4+22y^2z^2-4z^4\right)}{572}$$