PG No. 19 C_{3v} $3m$ (3m1 setting) [trigonal] (polar, internal polar dipole)

* Harmonics for rank 0

$\vec{\mathbb{Q}}_{0}^{(1,1)}[q](A_1)$

** symmetry

$$1$$

** expression

$$\frac{\sqrt{3}Q_x x}{3} + \frac{\sqrt{3}Q_y y}{3} + \frac{\sqrt{3}Q_z z}{3}$$

* Harmonics for rank 1

$\vec{\mathbb{Q}}_{1}^{(1,-1)}[q](A_1)$

** symmetry

$$z$$

** expression

$$Q_z$$

$\vec{\mathbb{Q}}_{1}^{(1,1)}[q](A_1)$

** symmetry

$$z$$

** expression

$$\frac{3\sqrt{10}Q_x xz}{10} + \frac{3\sqrt{10}Q_y yz}{10} - \frac{\sqrt{10}Q_z\left(x^2+y^2-2z^2\right)}{10}$$

$\vec{\mathbb{Q}}_{1,1}^{(1,-1)}[q](E)$, $\vec{\mathbb{Q}}_{1,2}^{(1,-1)}[q](E)$

** symmetry

$$x$$

$$y$$

** expression

$$Q_x$$

$$Q_y$$

$\vec{\mathbb{Q}}_{1,1}^{(1,1)}[q](E)$, $\vec{\mathbb{Q}}_{1,2}^{(1,1)}[q](E)$

** symmetry

$$x$$

$$y$$

** expression

$$\frac{\sqrt{10}Q_x\left(2x^2-y^2-z^2\right)}{10} + \frac{3\sqrt{10}Q_y xy}{10} + \frac{3\sqrt{10}Q_z xz}{10}$$

$$\frac{3\sqrt{10}Q_x xy}{10} - \frac{\sqrt{10}Q_y\left(x^2-2y^2+z^2\right)}{10} + \frac{3\sqrt{10}Q_z yz}{10}$$

* Harmonics for rank 2

$\vec{\mathbb{Q}}_{2}^{(1,-1)}[q](A_1)$

** symmetry

$$-\frac{x^2}{2} - \frac{y^2}{2} + z^2$$

** expression

$$-\frac{\sqrt{6}Q_x x}{6} - \frac{\sqrt{6}Q_y y}{6} + \frac{\sqrt{6}Q_z z}{3}$$

$\vec{\mathbb{Q}}_{2}^{(1,1)}[q](A_1)$

** symmetry

$$-\frac{x^2}{2} - \frac{y^2}{2} + z^2$$

** expression

$$-\frac{\sqrt{21}Q_x x\left(x^2+y^2-4z^2\right)}{14}-\frac{\sqrt{21}Q_y y\left(x^2+y^2-4z^2\right)}{14}-\frac{\sqrt{21}Q_z z\left(3x^2+3y^2-2z^2\right)}{14}$$

$\vec{\mathbb{Q}}_{2,1}^{(1,-1)}[q](E,1)$, $\vec{\mathbb{Q}}_{2,2}^{(1,-1)}[q](E,1)$
** symmetry

$$\sqrt{3}xz$$

$$\sqrt{3}yz$$

** expression

$$\frac{\sqrt{2}Q_x z}{2}+\frac{\sqrt{2}Q_z x}{2}$$

$$\frac{\sqrt{2}Q_y z}{2}+\frac{\sqrt{2}Q_z y}{2}$$

$\vec{\mathbb{Q}}_{2,1}^{(1,-1)}[q](E,2)$, $\vec{\mathbb{Q}}_{2,2}^{(1,-1)}[q](E,2)$
** symmetry

$$\sqrt{3}xy$$

$$\frac{\sqrt{3}\left(x-y\right)\left(x+y\right)}{2}$$

** expression

$$\frac{\sqrt{2}Q_x y}{2}+\frac{\sqrt{2}Q_y x}{2}$$

$$\frac{\sqrt{2}Q_x x}{2}-\frac{\sqrt{2}Q_y y}{2}$$

$\vec{\mathbb{Q}}_{2,1}^{(1,1)}[q](E,1)$, $\vec{\mathbb{Q}}_{2,2}^{(1,1)}[q](E,1)$
** symmetry

$$\sqrt{3}xz$$

$$\sqrt{3}yz$$

** expression

$$\frac{\sqrt{7}Q_x z\left(4x^2-y^2-z^2\right)}{7}+\frac{5\sqrt{7}Q_y xyz}{7}-\frac{\sqrt{7}Q_z x\left(x^2+y^2-4z^2\right)}{7}$$

$$\frac{5\sqrt{7}Q_x xyz}{7}-\frac{\sqrt{7}Q_y z\left(x^2-4y^2+z^2\right)}{7}-\frac{\sqrt{7}Q_z y\left(x^2+y^2-4z^2\right)}{7}$$

$\vec{\mathbb{Q}}_{2,1}^{(1,1)}[q](E,2)$, $\vec{\mathbb{Q}}_{2,2}^{(1,1)}[q](E,2)$
** symmetry

$$\sqrt{3}xy$$

$$\frac{\sqrt{3}\left(x-y\right)\left(x+y\right)}{2}$$

** expression

$$\frac{\sqrt{7}Q_x y\left(4x^2-y^2-z^2\right)}{7}-\frac{\sqrt{7}Q_y x\left(x^2-4y^2+z^2\right)}{7}+\frac{5\sqrt{7}Q_z xyz}{7}$$

$$\frac{\sqrt{7}Q_x x\left(3x^2-7y^2-2z^2\right)}{14}+\frac{\sqrt{7}Q_y y\left(7x^2-3y^2+2z^2\right)}{14}+\frac{5\sqrt{7}Q_z z\left(x-y\right)\left(x+y\right)}{14}$$

* Harmonics for rank 3
$\vec{\mathbb{Q}}_{3}^{(1,-1)}[q](A_1,1)$
** symmetry

$$-\frac{z\left(3x^2+3y^2-2z^2\right)}{2}$$

** expression

$$-\frac{\sqrt{15}Q_x xz}{5} - \frac{\sqrt{15}Q_y yz}{5} - \frac{\sqrt{15}Q_z\left(x^2+y^2-2z^2\right)}{10}$$

$\vec{\mathbb{Q}}_3^{(1,-1)}[q](A_1,2)$
** symmetry

$$\frac{\sqrt{10}y\left(3x^2-y^2\right)}{4}$$

** expression

$$\frac{\sqrt{6}Q_x xy}{2} + \frac{\sqrt{6}Q_y\left(x-y\right)\left(x+y\right)}{4}$$

$\vec{\mathbb{Q}}_3^{(1,1)}[q](A_1,1)$
** symmetry

$$-\frac{z\left(3x^2+3y^2-2z^2\right)}{2}$$

** expression

$$-\frac{5Q_x xz\left(3x^2+3y^2-4z^2\right)}{12} - \frac{5Q_y yz\left(3x^2+3y^2-4z^2\right)}{12} + \frac{Q_z\left(3x^4+6x^2y^2-24x^2z^2+3y^4-24y^2z^2+8z^4\right)}{12}$$

$\vec{\mathbb{Q}}_3^{(1,1)}[q](A_1,2)$
** symmetry

$$\frac{\sqrt{10}y\left(3x^2-y^2\right)}{4}$$

** expression

$$\frac{\sqrt{10}Q_x xy\left(15x^2-13y^2-6z^2\right)}{24} - \frac{\sqrt{10}Q_y\left(3x^4-21x^2y^2+3x^2z^2+4y^4-3y^2z^2\right)}{24} + \frac{7\sqrt{10}Q_z yz\left(3x^2-y^2\right)}{24}$$

$\vec{\mathbb{Q}}_3^{(1,-1)}[q](A_2)$
** symmetry

$$\frac{\sqrt{10}x\left(x^2-3y^2\right)}{4}$$

** expression

$$\frac{\sqrt{6}Q_x\left(x-y\right)\left(x+y\right)}{4} - \frac{\sqrt{6}Q_y xy}{2}$$

$\vec{\mathbb{Q}}_3^{(1,1)}[q](A_2)$
** symmetry

$$\frac{\sqrt{10}x\left(x^2-3y^2\right)}{4}$$

** expression

$$\frac{\sqrt{10}Q_x\left(4x^4-21x^2y^2-3x^2z^2+3y^4+3y^2z^2\right)}{24} + \frac{\sqrt{10}Q_y xy\left(13x^2-15y^2+6z^2\right)}{24} + \frac{7\sqrt{10}Q_z xz\left(x^2-3y^2\right)}{24}$$

$\vec{\mathbb{Q}}_{3,1}^{(1,-1)}[q](E,1)$, $\vec{\mathbb{Q}}_{3,2}^{(1,-1)}[q](E,1)$
** symmetry

$$-\frac{\sqrt{6}x\left(x^2+y^2-4z^2\right)}{4}$$

$$-\frac{\sqrt{6}y\left(x^2+y^2-4z^2\right)}{4}$$

** expression

$$-\frac{\sqrt{10}Q_x\left(3x^2+y^2-4z^2\right)}{20} - \frac{\sqrt{10}Q_y xy}{10} + \frac{2\sqrt{10}Q_z xz}{5}$$

$$-\frac{\sqrt{10}Q_x xy}{10} - \frac{\sqrt{10}Q_y\left(x^2+3y^2-4z^2\right)}{20} + \frac{2\sqrt{10}Q_z yz}{5}$$

$\vec{\mathbb{Q}}_{3,1}^{(1,-1)}[q](E,2)$, $\vec{\mathbb{Q}}_{3,2}^{(1,-1)}[q](E,2)$

** symmetry

$$\sqrt{15}xyz$$

$$\frac{\sqrt{15}z\left(x-y\right)\left(x+y\right)}{2}$$

** expression

$$Q_{x}yz+Q_{y}xz+Q_{z}xy$$

$$Q_{x}xz-Q_{y}yz+\frac{Q_{z}\left(x-y\right)\left(x+y\right)}{2}$$

$\vec{\mathbb{Q}}_{3,1}^{(1,1)}[q](E,1)$, $\vec{\mathbb{Q}}_{3,2}^{(1,1)}[q](E,1)$

** symmetry

$$-\frac{\sqrt{6}x\left(x^{2}+y^{2}-4z^{2}\right)}{4}$$

$$-\frac{\sqrt{6}y\left(x^{2}+y^{2}-4z^{2}\right)}{4}$$

** expression

$$-\frac{\sqrt{6}Q_{x}\left(4x^{4}+3x^{2}y^{2}-27x^{2}z^{2}-y^{4}+3y^{2}z^{2}+4z^{4}\right)}{24}-\frac{5\sqrt{6}Q_{y}xy\left(x^{2}+y^{2}-6z^{2}\right)}{24}-\frac{5\sqrt{6}Q_{z}xz\left(3x^{2}+3y^{2}-4z^{2}\right)}{24}$$

$$-\frac{5\sqrt{6}Q_{x}xy\left(x^{2}+y^{2}-6z^{2}\right)}{24}+\frac{\sqrt{6}Q_{y}\left(x^{4}-3x^{2}y^{2}-3x^{2}z^{2}-4y^{4}+27y^{2}z^{2}-4z^{4}\right)}{24}-\frac{5\sqrt{6}Q_{z}yz\left(3x^{2}+3y^{2}-4z^{2}\right)}{24}$$

$\vec{\mathbb{Q}}_{3,1}^{(1,1)}[q](E,2)$, $\vec{\mathbb{Q}}_{3,2}^{(1,1)}[q](E,2)$

** symmetry

$$\sqrt{15}xyz$$

$$\frac{\sqrt{15}z\left(x-y\right)\left(x+y\right)}{2}$$

** expression

$$\frac{\sqrt{15}Q_{x}yz\left(6x^{2}-y^{2}-z^{2}\right)}{6}-\frac{\sqrt{15}Q_{y}xz\left(x^{2}-6y^{2}+z^{2}\right)}{6}-\frac{\sqrt{15}Q_{z}xy\left(x^{2}+y^{2}-6z^{2}\right)}{6}$$

$$\frac{\sqrt{15}Q_{x}xz\left(5x^{2}-9y^{2}-2z^{2}\right)}{12}+\frac{\sqrt{15}Q_{y}yz\left(9x^{2}-5y^{2}+2z^{2}\right)}{12}-\frac{\sqrt{15}Q_{z}\left(x-y\right)\left(x+y\right)\left(x^{2}+y^{2}-6z^{2}\right)}{12}$$

* Harmonics for rank 4

$\vec{\mathbb{Q}}_{4}^{(1,-1)}[q](A_{1},1)$

** symmetry

$$\frac{3x^{4}}{8}+\frac{3x^{2}y^{2}}{4}-3x^{2}z^{2}+\frac{3y^{4}}{8}-3y^{2}z^{2}+z^{4}$$

** expression

$$\frac{3\sqrt{7}Q_{x}x\left(x^{2}+y^{2}-4z^{2}\right)}{28}+\frac{3\sqrt{7}Q_{y}y\left(x^{2}+y^{2}-4z^{2}\right)}{28}-\frac{\sqrt{7}Q_{z}z\left(3x^{2}+3y^{2}-2z^{2}\right)}{7}$$

$\vec{\mathbb{Q}}_{4}^{(1,-1)}[q](A_{1},2)$

** symmetry

$$\frac{\sqrt{70}yz\left(3x^{2}-y^{2}\right)}{4}$$

** expression

$$\frac{3\sqrt{10}Q_{x}xyz}{4}+\frac{3\sqrt{10}Q_{y}z\left(x-y\right)\left(x+y\right)}{8}+\frac{\sqrt{10}Q_{z}y\left(3x^{2}-y^{2}\right)}{8}$$

$\vec{\mathbb{Q}}_{4}^{(1,1)}[q](A_{1},1)$

** symmetry

$$\frac{3x^{4}}{8}+\frac{3x^{2}y^{2}}{4}-3x^{2}z^{2}+\frac{3y^{4}}{8}-3y^{2}z^{2}+z^{4}$$

** expression

$$\frac{3\sqrt{55}Q_x x\left(x^4+2x^2y^2-12x^2z^2+y^4-12y^2z^2+8z^4\right)}{88}+\frac{3\sqrt{55}Q_y y\left(x^4+2x^2y^2-12x^2z^2+y^4-12y^2z^2+8z^4\right)}{88}+\frac{\sqrt{55}Q_z z\left(15x^4+30x^2y^2-40x^2z^2+15y^4-40y^2z^2+8z^4\right)}{88}$$

$\vec{\mathbb{Q}}_4^{(1,1)}[q](A_1,2)$

** symmetry

$$\frac{\sqrt{70}yz\left(3x^2-y^2\right)}{4}$$

** expression

$$\frac{3\sqrt{154}Q_x xyz\left(7x^2-5y^2-2z^2\right)}{44}-\frac{3\sqrt{154}Q_y z\left(x^4-9x^2y^2+x^2z^2+2y^4-y^2z^2\right)}{44}-\frac{\sqrt{154}Q_z y\left(3x^2-y^2\right)\left(x^2+y^2-8z^2\right)}{44}$$

$\vec{\mathbb{Q}}_4^{(1,-1)}[q](A_2)$

** symmetry

$$\frac{\sqrt{70}xz\left(x^2-3y^2\right)}{4}$$

** expression

$$\frac{3\sqrt{10}Q_x z\left(x-y\right)\left(x+y\right)}{8}-\frac{3\sqrt{10}Q_y xyz}{4}+\frac{\sqrt{10}Q_z x\left(x^2-3y^2\right)}{8}$$

$\vec{\mathbb{Q}}_4^{(1,1)}[q](A_2)$

** symmetry

$$\frac{\sqrt{70}xz\left(x^2-3y^2\right)}{4}$$

** expression

$$\frac{3\sqrt{154}Q_x z\left(2x^4-9x^2y^2-x^2z^2+y^4+y^2z^2\right)}{44}+\frac{3\sqrt{154}Q_y xyz\left(5x^2-7y^2+2z^2\right)}{44}-\frac{\sqrt{154}Q_z x\left(x^2-3y^2\right)\left(x^2+y^2-8z^2\right)}{44}$$

$\vec{\mathbb{Q}}_{4,1}^{(1,-1)}[q](E,1)$, $\vec{\mathbb{Q}}_{4,2}^{(1,-1)}[q](E,1)$

** symmetry

$$-\frac{\sqrt{10}xz\left(3x^2+3y^2-4z^2\right)}{4}$$

$$-\frac{\sqrt{10}yz\left(3x^2+3y^2-4z^2\right)}{4}$$

** expression

$$-\frac{\sqrt{70}Q_x z\left(9x^2+3y^2-4z^2\right)}{56}-\frac{3\sqrt{70}Q_y xyz}{28}-\frac{3\sqrt{70}Q_z x\left(x^2+y^2-4z^2\right)}{56}$$

$$-\frac{3\sqrt{70}Q_x xyz}{28}-\frac{\sqrt{70}Q_y z\left(3x^2+9y^2-4z^2\right)}{56}-\frac{3\sqrt{70}Q_z y\left(x^2+y^2-4z^2\right)}{56}$$

$\vec{\mathbb{Q}}_{4,1}^{(1,-1)}[q](E,2)$, $\vec{\mathbb{Q}}_{4,2}^{(1,-1)}[q](E,2)$

** symmetry

$$-\frac{\sqrt{35}xy\left(x-y\right)\left(x+y\right)}{2}$$

$$\frac{\sqrt{35}\left(x^2-2xy-y^2\right)\left(x^2+2xy-y^2\right)}{8}$$

** expression

$$-\frac{\sqrt{5}Q_x y\left(3x^2-y^2\right)}{4}-\frac{\sqrt{5}Q_y x\left(x^2-3y^2\right)}{4}$$

$$\frac{\sqrt{5}Q_x x\left(x^2-3y^2\right)}{4}-\frac{\sqrt{5}Q_y y\left(3x^2-y^2\right)}{4}$$

$\vec{\mathbb{Q}}_{4,1}^{(1,-1)}[q](E,3)$, $\vec{\mathbb{Q}}_{4,2}^{(1,-1)}[q](E,3)$

** symmetry

$$-\frac{\sqrt{5}xy\left(x^2+y^2-6z^2\right)}{2}$$

$$-\frac{\sqrt{5}\left(x-y\right)\left(x+y\right)\left(x^{2}+y^{2}-6z^{2}\right)}{4}$$

** expression

$$-\frac{\sqrt{35}Q_{x}y\left(3x^{2}+y^{2}-6z^{2}\right)}{28}-\frac{\sqrt{35}Q_{y}x\left(x^{2}+3y^{2}-6z^{2}\right)}{28}+\frac{3\sqrt{35}Q_{z}xyz}{7}$$

$$-\frac{\sqrt{35}Q_{x}x\left(x^{2}-3z^{2}\right)}{14}+\frac{\sqrt{35}Q_{y}y\left(y^{2}-3z^{2}\right)}{14}+\frac{3\sqrt{35}Q_{z}z\left(x-y\right)\left(x+y\right)}{14}$$

$\vec{\mathbb{Q}}_{4,1}^{(1,1)}[q](E,1)$, $\vec{\mathbb{Q}}_{4,2}^{(1,1)}[q](E,1)$

** symmetry

$$-\frac{\sqrt{10}xz\left(3x^{2}+3y^{2}-4z^{2}\right)}{4}$$

$$-\frac{\sqrt{10}yz\left(3x^{2}+3y^{2}-4z^{2}\right)}{4}$$

** expression

$$-\frac{\sqrt{22}Q_{x}z\left(18x^{4}+15x^{2}y^{2}-41x^{2}z^{2}-3y^{4}+y^{2}z^{2}+4z^{4}\right)}{44}-\frac{21\sqrt{22}Q_{y}xyz\left(x^{2}+y^{2}-2z^{2}\right)}{44}+\frac{3\sqrt{22}Q_{z}x\left(x^{4}+2x^{2}y^{2}-12x^{2}z^{2}+y^{4}-12y^{2}z^{2}+8z^{4}\right)}{44}$$

$$-\frac{21\sqrt{22}Q_{x}xyz\left(x^{2}+y^{2}-2z^{2}\right)}{44}+\frac{\sqrt{22}Q_{y}z\left(3x^{4}-15x^{2}y^{2}-x^{2}z^{2}-18y^{4}+41y^{2}z^{2}-4z^{4}\right)}{44}+\frac{3\sqrt{22}Q_{z}y\left(x^{4}+2x^{2}y^{2}-12x^{2}z^{2}+y^{4}-12y^{2}z^{2}+8z^{4}\right)}{44}$$

$\vec{\mathbb{Q}}_{4,1}^{(1,1)}[q](E,2)$, $\vec{\mathbb{Q}}_{4,2}^{(1,1)}[q](E,2)$

** symmetry

$$-\frac{\sqrt{35}xy\left(x-y\right)\left(x+y\right)}{2}$$

$$\frac{\sqrt{35}\left(x^{2}-2xy-y^{2}\right)\left(x^{2}+2xy-y^{2}\right)}{8}$$

** expression

$$-\frac{\sqrt{77}Q_{x}y\left(6x^{4}-11x^{2}y^{2}-3x^{2}z^{2}+y^{4}+y^{2}z^{2}\right)}{22}+\frac{\sqrt{77}Q_{y}x\left(x^{4}-11x^{2}y^{2}+x^{2}z^{2}+6y^{4}-3y^{2}z^{2}\right)}{22}-\frac{9\sqrt{77}Q_{z}xyz\left(x-y\right)\left(x+y\right)}{22}$$

$$\frac{\sqrt{77}Q_{x}x\left(5x^{4}-46x^{2}y^{2}-4x^{2}z^{2}+21y^{4}+12y^{2}z^{2}\right)}{88}+\frac{\sqrt{77}Q_{y}y\left(21x^{4}-46x^{2}y^{2}+12x^{2}z^{2}+5y^{4}-4y^{2}z^{2}\right)}{88}+\frac{9\sqrt{77}Q_{z}z\left(x^{2}-2xy-y^{2}\right)\left(x^{2}+2xy-y^{2}\right)}{88}$$

$\vec{\mathbb{Q}}_{4,1}^{(1,1)}[q](E,3)$, $\vec{\mathbb{Q}}_{4,2}^{(1,1)}[q](E,3)$

** symmetry

$$-\frac{\sqrt{5}xy\left(x^{2}+y^{2}-6z^{2}\right)}{2}$$

$$-\frac{\sqrt{5}\left(x-y\right)\left(x+y\right)\left(x^{2}+y^{2}-6z^{2}\right)}{4}$$

** expression

$$-\frac{\sqrt{11}Q_{x}y\left(6x^{4}+5x^{2}y^{2}-51x^{2}z^{2}-y^{4}+5y^{2}z^{2}+6z^{4}\right)}{22}+\frac{\sqrt{11}Q_{y}x\left(x^{4}-5x^{2}y^{2}-5x^{2}z^{2}-6y^{4}+51y^{2}z^{2}-6z^{4}\right)}{22}-\frac{21\sqrt{11}Q_{z}xyz\left(x^{2}+y^{2}-2z^{2}\right)}{22}$$

$$-\frac{\sqrt{11}Q_{x}x\left(5x^{4}-4x^{2}y^{2}-46x^{2}z^{2}-9y^{4}+66y^{2}z^{2}+12z^{4}\right)}{44}-\frac{\sqrt{11}Q_{y}y\left(9x^{4}+4x^{2}y^{2}-66x^{2}z^{2}-5y^{4}+46y^{2}z^{2}-12z^{4}\right)}{44}-\frac{21\sqrt{11}Q_{z}z\left(x-y\right)\left(x+y\right)\left(x^{2}+y^{2}-2z^{2}\right)}{44}$$